CONNECTED SUMS OF CLOSED ORIENTABLE TRIANGULATED 3-MANIFOLDS

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ABSTRACT. In this paper, we describe geometrical constructions to obtain triangulations of connected sums of closed orientable triangulated 3-manifolds. Using these constructions, we show that it takes time polynomial in the number of tetrahedra to check if a closed orientable 3-manifold, equipped with a minimal triangulation, is reducible or not. This result can easily be generalized to compact orientable 3-manifolds with non-empty boundary.

1. INTRODUCTION

1.1. **Normal Surfaces.** This paper relies heavily on the process of "collapsing" normal 2-spheres, which was first described by Jaco and Rubinstein in [9]. In section 2, we describe an alternate "collapsing" process, based on the existence of *collapsing surfaces*. In section 3, we give a proof of the main theorem:

Theorem 2.1: *Let M be a closed orientable triangulated 3-manifold with t tetrahedra. Let S be a non-trivial normal 2-sphere. Then M is homeomorphic to $M_1 \# M_2 ... \# M_k$ $\# r_1(S^1 \times S^2) \# r_2 \mathbf{RP}^3 \# r_3 L(3,1)$, where r_1, r_2, r_3, $k \geq 0$, $|M_1| + ... + |M_k| < |M|$ and the M_i's are closed orientable triangulated 3-manifolds.*

In section 4, we describe geometrical constructions to obtain triangulations for connected sums of closed orientable triangulated 3-manifolds. In section 5, we prove our main result regarding the existence of normal 2-spheres with at most 2 non-zero quadrilateral types in minimal triangulations. Finally, in the last section, we fully describe an algorithm, due to Andrew Casson, to decompose a closed orientable triangulated 3-manifold into irreducible pieces. Using this algorithm, we show how one can check, in time polynomial with respect to the number of tetrahedra in the triangulation, if a closed orientable 3-manifold equipped with a minimal triangulation is reducible or not.

All the 3-manifolds considered in this paper are in the piecewise linear category, i.e. every 3-manifold will be associated with a triangulation.

1.1. **Definition.** A **pseudo-triangulation** of a compact orientable 3-manifold M is a set Δ of pairwise disjoint tetrahedra, together with a family of homeomorphisms, Φ, where the domain and image of each homeomorphism consist of faces of tetrahedra. The identification space, Δ/Φ, is homeomorphic to M. If Δ consist of n tetrahedra, we write $|\mathbf{M}| = n$. A **genuine triangulation** is a pseudo-triangulation such that every tetrahedron is embedded in M. By abuse of language, we call a pseudo-triangulation a triangulation. Moreover, M will always be associated with a fixed triangulation.

We denote the i-skeleton of M by $T^{(i)}$, for $i = 0, 1, 2, 3$. Let $R : M \times I \to M$ be an isotopy of M. R is called a **normal isotopy** if it is invariant in each tetrahedron Δ_i, i.e. $R(\Delta_i, t) = \Delta_i$ for all $t \in I$. **Normal surfaces**, which were first developed by W.Haken [5], are embedded surfaces in M which intersect each tetrahedron in planes in general position with respect to $T^{(2)}$. There are 7 different isotopy classes of planes (called **elementary disks**) for each tetrahedron: 4 triangle types and 3 quadrilateral types. A normal surface

can thus be described as an ordered set of elementary disks in each tetrahedron. Using this definition, a normal surface describes a whole class of embedded surfaces which are all normal isotopic. Hence, by a normal surface we mean its normal isotopy class.

We call a normal 2-sphere *trivial* if it intersects the tetrahedra in triangles only, and we call it non-trivial otherwise.

Given a closed embedded normal surface S, it has to satisfy the *quadrilateral property*: if there is a quadrilateral type in a tetrahedron, then no other types of quadrilateral can exist in that same tetrahedron. Moreover, F must satisfy *matching equations*:

If M has t tetrahedra, there are exactly $6t$ matchings to be satisfied, 3 for each face of the triangulation. We have noticed above that a normal surface is described by a set of $7t$ elementary disks, 7 for each tetrahedron. So one way to think of a normal surface is to look at it as a non-negative integer valued vector with $7t$ entries satisfying the quadrilateral property and the matching equations. Given a normal surface F, we may refer to it as x_F, where $x_F = (x_1, x_2, \ldots, x_{7t})$. The matching equations may be seen as a set of $6t$ equations of the form $x_i + x_j = x_k + x_l$.

Let F and G be two normal surfaces which intersect each other. Let x_F and x_G be their corresponding vectors. Suppose that F and G intersect each other with the only requirement that their quadrilateral types are the same in each tetrahedron. We can perform an operation, called a *regular exchange*, described in one of many ways in figure 1 below, such that the resulting pieces are disjoint elementary disks.
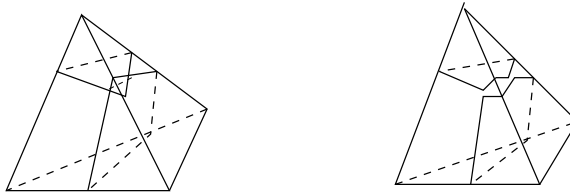

FIGURE 1. Before and after a regular exchange.

The surface obtained in this manner is the normal surface represented by the vector $x_F + x_G$. This operation is called the *surface addition* or *Haken sum* of F and G, and it is unique up to isotopy. We define the *weight* of a normal surface F, denoted by *wt(F)*, as the number of intersection points of F with $T^{(1)}$. It is well known that the Haken sum preserves the weight and the Euler characteristic of normal surfaces.

A normal surface is called *fundamental* if it cannot be written as the sum of two non-isotopically parallel surfaces. Obviously, a fundamental surface has to be connected. Haken proved that the set of fundamental surfaces is finite and that they can be found algorithmically (see [4] and [5]).

Consider the set of non-negative real solutions to the matching equations and satisfying the quadrilateral property. It is well known, through linear programming, that this set forms a cone in $\mathbf{R}^{7t}$. We intersect this cone with the set of solutions to the equation: $\sum_{i=1}^{7t} x_i = 1$. We obtain a convex polyhedron called the *projective solution space* of M with respect to its triangulation T, and we denote it by $P(M, T)$. For each normal surface S, there corresponds a rational vector $\bar{S}$ in $P(M, T)$ called the projective class of S. Conversely, any rational vector in $P(M, T)$ can be multiplied by an integer to obtain a vector representing a normal surface.

It can be shown that the vertices of $P(M, T)$ have rational entries. Let v be a vertex of $P(M, T)$ and let k be the smallest integer such that $k \cdot v$ is an integral solution. We call $k \cdot v$ a *vertex solution*. In particular, an integral solution F is a vertex solution if and only if the integral solutions, X and Y, to the equation $n \cdot F = X + Y$ are multiples

of F. We call F a **vertex surface** if it is connected, 2-sided, and if its representative on $P(M, T)$ is a vertex. Note, if F is a vertex surface, then either F is also a vertex solution or it is the double of a vertex solution. See [11], [5], [6], [9], [10], and [14] for more details on normal surfaces.

2. COLLAPSING NORMAL 2-SPHERES

This work is directly inspired by a theorem, stated first by Jaco and Rubinstein, which appeared in [9]. The original theorem is the following:

Theorem (Jaco-Rubinstein): Let M be a closed orientable triangulated 3-manifold. If M contains a non-trivial 2-sphere, then either $M \cong M_1 \# M_2$, with $|M_1| + |M_2| < |M|$, or $M \cong M_1 \# r_1(S^1 \times S^2) \# r_2 \mathbf{RP}^3 \# r_3 L(3, 1)$, with $|M_1| < |M|$.

In this section, we define the necessary terminology to prove the following theorem.

2.1. Theorem. *Let M be a closed orientable triangulated 3-manifold with t tetrahedra. Let S be a non-trivial normal 2-sphere. Then M is homeomorphic to $M_1 \# M_2 ... \# M_k$ $\# r_1(S^1 \times S^2) \# r_2 \mathbf{RP}^3 \# r_3 L(3, 1)$, where $r_1, r_2, r_3, k \geq 0$, $|M_1| + ... + |M_k| < |M|$ and the M_i's are closed orientable triangulated 3-manifolds.*

If we cut M along S, we obtain a cell decomposition of a 3-manifold $M \backslash S$ with two 2-spheres as boundary. The idea is to collapse each of the two 2-spheres to a point, and obtain a well-defined triangulation for the resulting 3-manifold $M \backslash S$. After cutting M along S, there are 7 different types of polyhedra in the cell decomposition of $M \backslash S$ to consider: tetrahedra, truncated tetrahedra (with 1, 2, 3 or 4 truncations), prisms, truncated prisms (with 1 or 2 truncations), tips, $I \times$ quadrilateral, and $I \times$ triangles. The last two polyhedra are called ***I-bundles***. Some types of polyhedra may be combinatorially equivalent (e.g. a tetrahedron and a tip), but for topological reasons we consider them as different.
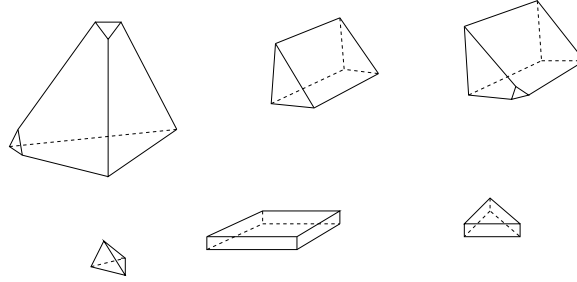

FIGURE 2. Six types of polyhedra in $M \backslash S$

2.2. Definition. Let M be a closed orientable triangulated 3-manifold and let S be an embedded orientable surface. We define $\mathbf{M} \backslash \mathbf{S}$ to be the 3-manifold $\overline{M - Nbhd(S)}$, where $Nbhd(S)$ denotes a regular neighborhood of S. By **collapsing** S in M we mean collapsing the two copies of S in $M \backslash S$ to points. If S is a 2-sphere, then collapsing S is topologically equivalent to cutting M along S and capping off with 3-balls.

From now on, S denotes a non-trivial normal 2-sphere unless otherwise stated. We now describe the process of collapsing a prism. To do so, we define a prism P as the quotient space $(I \times J) \times K/(a, 1, c_1) \sim (a, 1, c_2)$, where I, J, and K are unit intervals. Leaves of P correspond to sets (a, b, K) for fixed $a \in I$, $b \in J$, and $b \neq 0$. A similar foliation can be defined for truncated prisms.
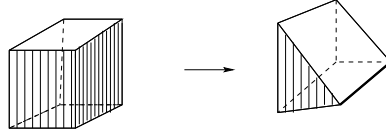
FIGURE 3. Induced foliation of a prism

Collapsing a (truncated) prism P means taking the quotient space $P/\sim$, where $x \sim y$ for any two points , x and y, belonging to the same leaf.

We call the **sides** of an I-bundle or a tip, the faces which were originally subsets of the 2-skeleton of M. We call the **face(s)** of an I-bundle or a tip, the face(s) which were originally embedded in S.

Similarly, we define the **sides**, the **top** face, and the **bottom** face of a (truncated) prism P. Our point here is to give a name for the two hexagonal faces of a truncated prism. It doesn't matter which is the top and which is the bottom.
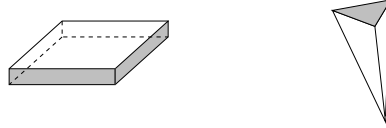

FIGURE 4. The sides of an I-bundle and the face of a tip

FIGURE 5. A side and the top of a (truncated) prism

A polyhedra P_1 is **adjacent** to a (truncated) prism P_2 if a side of P_2 coincides with a side of P_1. To each prism, there are at most 2 adjacent polyhedra.

A **collapsing annulus** is an annulus A embedded in the 2-skeleton of M with the following properties:
  - $\partial A = \beta_1 \cup \beta_2$, where β_1 and β_2 are composed of parallel normal arcs.
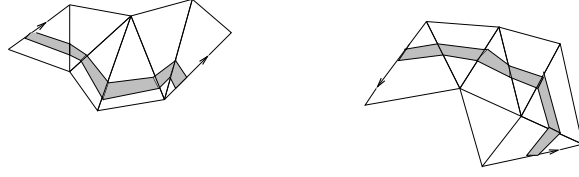 - $A \cap S = \partial A$.

A **collapsing Mobius band** is a Mobius band B embedded in the 2-skeleton of M with the following properties:
  - $\partial B = \beta$, where β is composed of pairs of parallel normal arcs.
  - $B \cap S = \partial B$.

A **collapsing disk** is a disk D embedded in the 2-skeleton of M with the following properties:
  - ∂D is composed of normal arcs and $D \cap M$ is composed of triangles only.
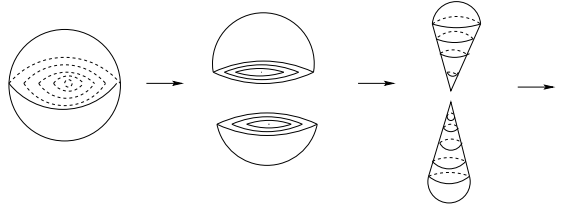 - $D \cap S = \partial D$.

Remark. Let S be a non-trivial normal 2-sphere in M, and let A be a collapsing annulus. A inherits a natural trivial I-bundle structure $S^1 \times I$. Let D_1 and D_2 be the two disjoint disks on S such that $\partial A = \partial D_1 \cup \partial D_2$. Consider the 3-manifold $M' = M\backslash(S \cup A)$. The boundary of M' consist of the union of 2 copies of A, A_1 and A_2, and 2 (possibly

FIGURE 6. A collapsing annulus and a collapsing Mobius band in $T^{(2)}$.

disconnected) copies of S, S_1 and S_2. By **collapsing** A in M', we mean taking the quotient space $M'/(\phi_1, \phi_2, \psi)$ where ϕ_i is the natural retraction on A_i, $\phi_i : S^1 \times$ I $\to \{$pt$\}$ $\times I$, and where ψ maps any connected component of S_i to a point. Note, collapsing A in M' is topologically equivalent to collapsing two disjoint 2-spheres in M, parallel to S and $A \cup D_1 \cup D_2$.

A collapsing annulus A is **inessential** if the 2-sphere $D_1 \cup D_2 \cup A$ bounds a ball B^3 such that $int(B^3) \cap S = \emptyset$. Otherwise, A is called essential. Note, if A is inessential, then collapsing A is topologically equivalent to collapsing S only.

Let S be a non-trivial normal 2-sphere in M, and let D be a collapsing disk. D inherits a natural foliation as in figure 7 below. Let D_1 and D_2 be the two disks on S, with disjoint interior, such that $\partial D = \partial D_1 = \partial D_2$. Consider the 3-manifold $M' = M\backslash(S \cup D)$. The boundary of M' consists of the union of 2 copies of D, D' and D", 1 connected copy of S, S_1 , and 1 disconnected copy of S, S_2 . By **collapsing** D in M', we mean taking the quotient space $M'/(\phi_1, \phi_2, \psi)$ where ϕ_1 (resp. ϕ_2) is the continuous map which maps each leaf of D' (resp. D") to a point, and where ψ maps any connected component of S_i to a point. Note, collapsing D in M' is topologically equivalent to collapsing two disjoint 2-spheres in M parallel to S and to $D \cup D_1 \cup D_2$.

FIGURE 7. Collapsing of the disk D.

A collapsing disk D is **inessential** if one of the 2-spheres, $D_1 \cup D$ or $D_2 \cup D$, bounds a ball B^3 such that $int(B^3) \cap S = \emptyset$. Otherwise, D is called essential. Note, if D is inessential, then collapsing D is topologically equivalent to collapsing S only.

Let S be a non-trivial normal 2-sphere in M, and let B be a collapsing Mobius band. The boundary of B is homeomorphic to an annulus A which inherits a natural trivial I-bundle structure. Let D_1 and D_2 be the two disks on S, with disjoint interior, such that $\partial D_1 = \partial D_2 = \partial B$. Let $M' = M\backslash S \cup B$. The boundary of M' consists of the union of one copy of A, one connected copy of S, and one disconnected copy of S. By **collapsing** B in M', we mean taking the quotient space $M'/(\phi, \psi)$, where ψ maps any connected component of S to a point and ϕ is the natural retraction on A. Collapsing B in M' is topologically equivalent to collapsing two disjoint 2-spheres in M, one parallel to S and one parallel to $D_1 \cup D_2 \cup A$ (note, the two 2-spheres bound a twice punctured $\mathbf{R}P^3$).

A collapsing annulus, Mobius band, or disk is always defined with respect to an embedded normal surface. These collapsing surfaces will be a major ingredient in the proof of Theorem 2.1. We mention here that a very similar collapsing process has already been described by Jaco and Rubinstein in [9].

3. Proof of Theorem 2.1

We are now ready to prove Theorem 2.1. The proof of this theorem is presented in the form of a procedure which, given a normal 2-sphere S, finds triangulations of some connected summands of M. The number of resulting summands is completely determined by M and S. Hence, given two topologically parallel non-isotopic normal 2-spheres, we could obtain different decompositions of M.

Let us start by cutting M along S. We obtain a cell decomposition of $M \backslash S$ composed of the 7 types of polyhedra described earlier. We summarize the procedure: Let S be the non-trivial normal 2-sphere which we collapse. We get rid of all tips and I-bundles by collapsing some collapsing annuli and disks. Each maximal collection of adjacent tips contributes to an S^3 summand and each maximal collection of adjacent I-bundles contributes to either an S^3 or an $\mathbf{R}P^3$ summand. We then collapse the prisms one at a time. Each collapsing may or may not contribute to one $L(3,1)$, one or two $\mathbf{R}P^3$, or one S^3 summand. Finally, we collapse each truncated tetrahedra and we count the number of $S^1 \times S^2$.

Step 1: Consider a maximal collection $\mathcal{C}$ of adjacent tips. First, note that a tip can only be adjacent to another tip or a prism. The union of the faces of these tips represents a connected subsurface S' of S, and so $\mathcal{C}$ can be seen as the cone over S'. Call $\partial \mathcal{C}$ the union of the sides along which a tip in $\mathcal{C}$ is adjacent to a prism. By definition, $\partial \mathcal{C}$ is is made of a nonempty union of collapsing disks. We collapse each such disk. Note, the collapsing of each collapsing disk preserves the cone structure of $\mathcal{C}$. In particular, each connected boundary component of S' is being collapsed to a point, and so S' is being collapsed to a 2-sphere. Hence, for each maximal collection $\mathcal{C}$ of adjacent tips, we obtain a cone over a 2-sphere which is itself homeomorphic to a ball. It is now clear that each $\mathcal{C}$ contributes to a trivial summand in the decomposition of M.
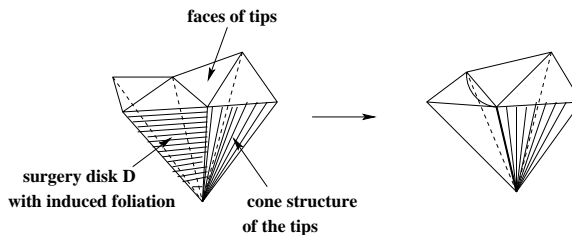

Figure 8. The collapsing of a collapsing disk along the side of a tip preserves the cone structure of $\mathcal{C}$.

Step 2: Consider a maximal collection $\mathcal{J}$ of adjacent I-bundles. Note, an I-bundle can only be adjacent to another I-bundle or to a truncated prism. The union of the faces of these I-bundles represents a possibly disconnected subsurface S' of S, and so $\mathcal{J}$ can be seen as an I-bundle over a surface B whose double cover is S'. Call $\partial \mathcal{J}$ the union of the sides along which an I-bundle in $\mathcal{J}$ is adjacent to a truncated prism. If $\partial \mathcal{J}$ is empty then $\mathcal{J}$ is an I-bundle whose boundary is S which means that $\mathcal{J}$ is homeomorphic to a punctured $\mathbf{R}P^3$. If $\partial \mathcal{J}$ is nonempty, then it consists of a union of collapsing annuli. We collapse each such annulus. Note, the collapsing of each annulus preserves the I-bundle

structure of $\mathcal{J}$. In particular, each connected boundary component of S' is being collapsed to a point, and so S' is being collapsed to either one or two copies of a 2-sphere depending on whether S' is disconnected or not. Hence, for each maximal collection $\mathcal{J}$ of adjacent prisms, we obtain an I-bundle whose boundary is either one or two copies of a 2-sphere. It is now clear that each $\mathcal{J}$ contributes to either a trivial summand or an $\mathbf{R}P^3$ summand in the decomposition of M.
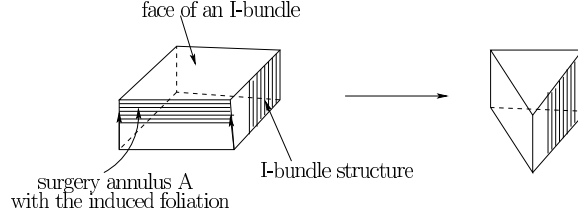

FIGURE 9. The collapsing of a collapsing annulus along the side of an I-bundle preserves the I-bundle structure of $\mathcal{J}$.

Remark. Let P be a (truncated) prism adjacent to a tip or an I-bundle Q. From step 1 and 2, we know that the common side F of P and Q is part of a collapsing annulus or disk A. Moreover, the collapsing of A induces a collapsing of F. Note, this collapsing of F coincides with our definition of the collapsing of P. Hence, after collapsing all the collapsing surfaces in step 1 and 2, we obtain a cell decomposition of $M \backslash S$ made of (truncated) tetrahedra, (truncated) prisms, and (truncated) prisms with their sides collapsed (these cells are called pillows in [9]). By abuse of language, we also call the latter cells prisms.

Step 3: This step consists of collapsing each (truncated) prism one at a time. Let P be a (truncated) prism. If P is embedded in $M \backslash S$, we can collapse it without changing the topology of $M \backslash S$. If P is not embedded, there are 4 cases to consider.
Case 1: All the leaves of one (or both) side of P are not embedded in M, and all the other leaves of P are. This happens if one (or both) side of P represents a collapsing annulus or disk depending on whether P is truncated or not. We cut along the collapsing surface and we collapse P.

Each side of P may be embedded but their union may represent a collapsing annulus or disk. In this case we cut along both sides and we collapse.
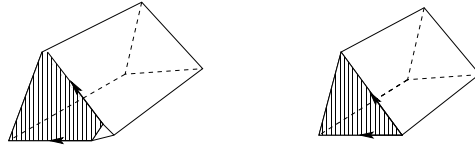

FIGURE 10. A collapsing annulus and a collapsing disk.

Case 2: Exactly one leaf in one (or both) side of P is not embedded, and all the other leaves are. This happens when one (or both) side of P represents a collapsing Mobius band B. We cut along B, we collapse P, and we add one (or two) $\mathbf{R}P^3$summand in the decomposition of M.

Each side of P may be embedded but their union may represent a collapsing Mobius band B. In this case, we cut along both sides, we collapse P, and we obtain one $\mathbf{R}P^3$ summand in the decomposition of M.

Case 3: None of the leaves of P are embedded. This happens when the top of P is identified to its bottom without any twist. In particular, both sides of P are collapsing annuli or disks depending on whether P is truncated or not. We collapse the collapsing surfaces. One can check that after the collapsing, P is homeomorphic to S^3. So we simply remove P from the cell decomposition of $M \backslash S$.
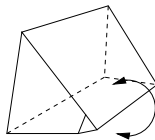

FIGURE 11. A prism with no embedded leaves.

Case 4: Exactly one leaf of P, which does not belong to one of its sides, is not embedded. This happens when the top and bottom of P are identified by a 1/3 twist. Here, P is homeomorphic to a solid torus T whose boundary consists of a collapsing annulus A and an annulus $A' \subset S$. Since A' lies on S both of its boundary components bound disjoint disks D_1 and D_2 on S. Moreover, both ∂D_1 and ∂D_2 represent a (3,1) curve (or (3, 2) curve) on ∂T. Hence $Nbhd(T \cup D_1 \cup D_2)$ is homeomorphic to a twice punctured lens space $L(3,1)$. We remove P from the cell decomposition of $M \backslash$. We then collapse A and we add one $L(3,1)$ summand in the decomposition of M.
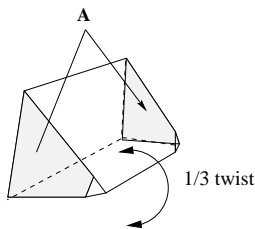

FIGURE 12. A truncated prism whose boundary is homeomorphic to a torus.
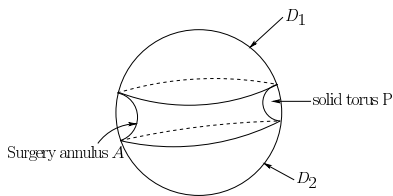

FIGURE 13. A twice punctured $L(3,1)$.

Suppose that the prism P was adjacent to another prism P'. Then the collapsing of P induced a collapsing of one (or both) side of P'. Since the collapsing of two adjacent

prisms coincide along their adjacent sides, P' is thought as a prism with one (or both) of its sides collapsed (these cells are called pillows in [9]). Treating each P' as such, we repeat step 3 for each prism (with or without their sides collapsed) found in the new cell decomposition of M.

Step 4: The only polyhedra left are tetrahedra and truncated tetrahedra. We collapse each truncated tetrahedron to a regular tetrahedron.

Step 5: We need to count the number of $S_1 \times S^2$ summands. Recall that collapsing a collapsing disk or annulus is topologically equivalent to collapsing a 2-sphere. Let n, m, and p be the respective number of collapsing annuli, disks and Möbius bands found in step 1, 2, and 3. Let k be the number of connected summands (including the trivial ones) of M obtained in step 1, 2, and 3. Then the number of $S^1 \times S^2$ summands is $n + m + p + 2 - k$. This comes from the fact that each non-separating 2-sphere contributes to one such summand. Indeed, if we have collapsed $n + m + p$ collapsing surfaces, we have actually collapsed $n + m + p + 1$ 2-spheres.

Note, if M_1, ..., M_k are the resulting triangulated summands of M, then it follows directly from the above construction that $|M_1| + ... + |M_k| < |M|$. In particular, if S has k non-zero quadrilateral types, then $|M_1| + ... + |M_k| = |M| - k$.

4. Connected sums of triangulated 3-manifolds

We describe here geometrical constructions to obtain a triangulation for the connected sums of closed orientable triangulated 3-manifolds. Intuitively, if one wants to take the connected sum of two such 3-manifolds, one can think of removing the interior of a tetrahedron in each manifold and glue the resulting manifolds along their boundaries. The problem is that the boundary of a tetrahedron may not be homeomorphic to a 2-sphere. One could always retriangulate a tetrahedron in each manifold to obtain tetrahedra with embedded boundary, but this construction seems to be artificial and is not as efficient as the one described here. Construction 4.1 is the only construction which is described thoroughly. As the reader will become more familiar with it, the other constructions are simple generalizations of the first one.

4.1. Construction. Let P and N be two triangulated closed orientable 3-manifolds with t_1 and t_2 tetrahedra, and v_1 and v_2 vertices, respectively. If not both of v_1 and v_2 are equal to 1, then there is a triangulation of $P \# N$ with $t_1 + t_2 + 2$ tetrahedra and $v_1 + v_2 - 2$ vertices. If $v_1 = v_2 = 1$, then there is a 1-vertex triangulation of $P \# N$ and $t_1 + t_2 + 4$ tetrahedra.

This triangulation of $P \# N$ does not have to be minimal, even if P and N are minimal. There are some cases, depending on the triangulations of P and N, where only $t_1 + t_2 + 1$ tetrahedra are needed to construct $P \# N$.

We assume that either P or N has more than one vertex. Indeed, let M be the one-vertex triangulation of a closed orientable 3-manifold which is the connected sum of exactly two irreducible 3-manifolds P and N, and let S be an essential normal 2-sphere. We collapse S. It is shown in Lemma 5.2 that one summand, say P, has one vertex (from the collapsing of S) and that N has two vertices (one from the collapsing of S and one from the original triangulation of M). Hence, it seems somewhat natural to make this assumption.

We first assume that v_1 and v_2 are both strictly greater than 1. Let a be a vertex of P and b a vertex of N. We remove a normal neighborhood of a and b. $P \backslash S_1$ and $N \backslash S_2$ are now 3-manifolds composed of tetrahedra and truncated tetrahedra with a boundary component being a triangulated 2-sphere, S_1 and S_2 respectively. We want to change the cell decompositions of $P \backslash S_1$ and $N \backslash S_2$ in order to glue the two manifolds along their boundary and obtain a well-defined triangulation for their connected sums. To simplify the notation, we will denote $P \backslash S_1$ and $N \backslash S_2$ by P' and N', respectively.

Because P and N have more than one vertex, we assume without loss of generality that a and b are chosen so that P' and N' contain a face of a truncated tetrahedron as in figure 14 below.
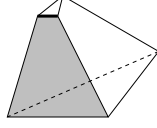

FIGURE 14. A truncated tetrahedron in P' and N'.

For future reference, we keep track of the above thickened edges in the triangulations of S_1 and S_2.

4.2. **Definition.** We say that a face of a tetrahedron in a triangulation is a **cone** if two of its edges are identified as in the figure 15 below. If, in addition to that, the endpoints of each edge are distinct, we call the cone a **good cone**.
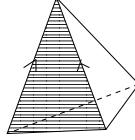

FIGURE 15. A cone in a tetrahedron.

Let τ be a triangulation of a surface S. The **dual** τ' of τ is a cell decomposition of S such that :

-There is a 1-1 correspondence between the i-simplices of τ and the $(2-i)$-simplices of τ'.

-Every triangle of τ (resp. every polygon of τ') contains exactly one vertex of τ' (resp. τ).

-Every edge of τ (reps. τ') intersects exactly one edge of τ' (resp. τ) in exactly one point.

One can check that for any fixed triangulation there is exactly one dual and vice versa. We denote by $\boldsymbol{G}$ the 3-valent planar graph made of the union of the vertices and edges of τ'. Let T_1 and T_2 be the triangulations corresponding to the 2-spheres S_1 and S_2, respectively. Let G_1 and G_2 be the respective graphs corresponding to the respective duals τ'_1 and τ'_2 of T_1 and T_2.

Suppose we have colored the vertices of τ'_1 and τ'_2. We say that G_1 and G_2 are **colored homeomorphic** (we write $G_1 = \overline{G_2}$) if there exists a cell preserving orientation reversing homeomorphism of the 2-sphere, sending τ'_1 to τ'_2 and sending each colored vertex of τ'_1 to a same color vertex in τ'_2. We describe a set of rules to transform the graphs G_1 and G_2 into graphs G'_1 and G'_2 such that G'_1 is colored homeomorphic to G'_2.

Let G_1 and G_2 be given. We first color the vertices of G_1 in white and the vertices of G_2 in black. We can add 4-valent red vertices on the original edges, and 3-valent black and white vertices and edges using the following rules:

(1) - Black vertices cannot be joined by an edge to white vertices.
(2) - Black (resp. white) vertices cannot be added to G_2 (resp. G_1). (This rule will be omitted when either P or N has a one-vertex triangulation).

(3) - Let r be a red vertex. Let e_1 and e_2 be opposite edges with one common end point r (since r is 4-valent, the notion of opposite edges is well-defined). Let a_1 and a_2 be the two other endpoints of e_1 and e_2, respectively. If a_1 is black (resp. white), then a_2 cannot be white (resp. black).

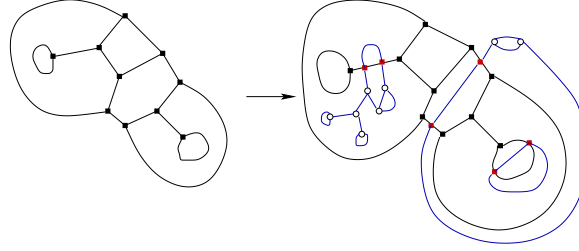(4) - The resulting graphs G'_1 and G'_2 must be connected and planar.

FIGURE 16. Example of a 3-valent graph G and another graph G' obtained by adding edges and vertices according to the above rules.

Let S_1 and S_2 be given. To the thickened edge in figure 14 corresponds a unique edge in the induced graphs G_1 and G_2, say e_1 and e_2 respectively. We insert two red vertices on each of these edges: r_1 and r'_1 on G_1, and r_2 and r'_2 on G_2. We then connect r_i and r'_i by an edge. Consider now a copy of the colored homeomorphic image of G_2 with the edge e_2 removed. We call this new graph $\overline{G_{2,e_2}}$. Color all the vertices of $\overline{G_{2,e_2}}$ in black. Draw an edge emanating from $\overline{r_1}$ and one emanating from r'_1. On these two edges, draw the graph corresponding to $\overline{G_{2,e_2}}$. We call the resulting graph G'_1. The same procedure can be done by adding $\overline{G_{1,e_1}}$ to G_2 to obtain the graph G'_2. One can check that G'_1 is colored homeomorphic to G'_2.
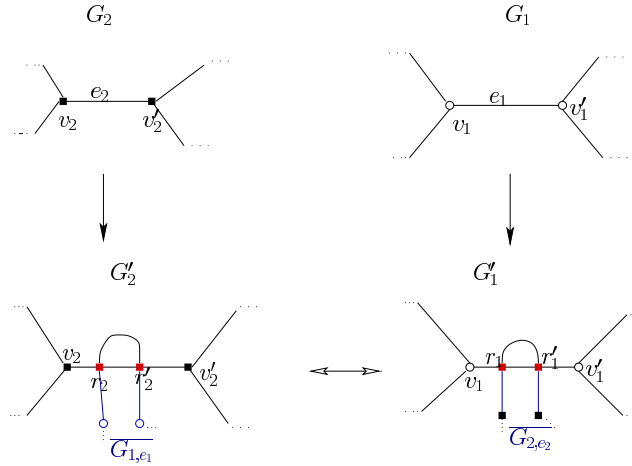

FIGURE 17. Colored homeomorphism between G'_1 and G'_2.

Let us take a closer look at the triangulations of S^1 and S^2. Note, because G'_1 and G'_2 are planar and connected, they represent the cell decompositions of 2-spheres. This

implies that their duals, S_1' and S_2', also represent the cell decompositions of 2-spheres. From the definition of a dual triangulation, each 3-valent (resp. 4-valent) vertex added on G_1 corresponds to a triangle (resp. quadrilateral) added on S_1, and each edge added on G_1 corresponds to an edge added on S_1. Hence, we changed the triangulation of S_1 and S_2 into cell decompositions of 2-spheres with triangles and quadrilaterals.
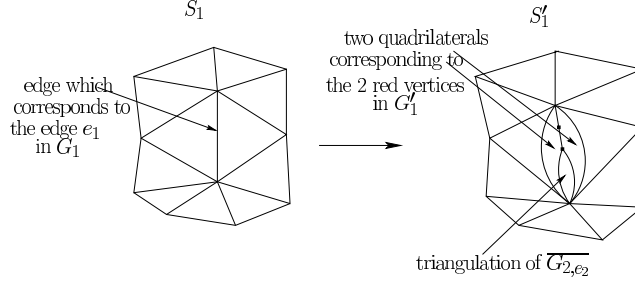

FIGURE 18. View of the new cell decomposition of $\partial P'$.

We saw that each 4-valent vertices added on G_1 corresponds to adding a quadrilateral on S_1. Here, adding a quadrilateral on S_1 is represented by inserting a prism in P', and adding a triangle on S_1 is represented by inserting the tip of a tetrahedron in P'. See figure 19 below.

As described above, the black vertices that we added on G_1, which correspond to tips of tetrahedra inserted in P', represent a graph isomorphic to $\overline{G_{2,e_2}}$. Since G_2 is the dual of the triangulation of a 2-sphere, $\overline{G_{2,e_2}}$ is the dual of the triangulation of a disk. Hence, the union of all the tips inserted in P' is homeomorphic to the cone over a disk, which is homeomorphic to a 3-ball. Hence, inserting the two prisms and the tips of tetrahedra did not change the topology of P' and N'. Moreover, every white (resp. black) triangle in S_1' (resp. S_2') corresponds to a truncated tetrahedron, and every white (resp. black) triangle in S_2' (resp. S_1') corresponds to the tip of a tetrahedra. If ϕ is the colored homeomorphism from G_1' to G_2', it is now clear that $(P \cup N)/\phi$ defines a well-defined triangulation for $P \# N$.

FIGURE 19. The new cell decomposition of P'.

Suppose now that N has more than one vertex and that P has exactly one vertex. Let G_1 and G_2 be the induced graphs of P and N respectively. Consider G_1. As before, we add two 4-valent vertices which correspond to inserting two truncated prisms. Because the prisms are truncated, each time we add a 4-valent vertex on G_1 we need to add a pair of white vertices too. Note, we are not given a choice on where to place this pair of

FIGURE 20. The connected sum of P and N.

white vertices. Note also that each black vertex added on G_1 corresponds to a truncated tip. Hence, when we add black vertices on G_1 corresponding to the graph $\overline{G_{2,e_2}}$, we also add white vertices on G_1 which correspond to the graph of G_{2,e_2}. After adding all the necessary vertices on G_1, we obtain a graph as in figure 21 below.
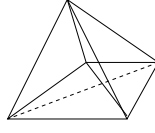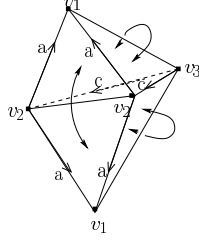

FIGURE 21. Colored homeomorphism between G_1' and G_2'.

Suppose now that both P and N have exactly one vertex in their triangulations. If we now remove a normal neighborhood of each vertex, we end up with a cell decomposition with no vertices which is impossible. On the other hand, we can take a subdivision of a tetrahedron of, say, P, to obtain a new triangulation of P with two vertices (see figure 22 below). We then apply the construction above to the new triangulation of P and to N. The triangulation of $P\#N$ has $|P_{new}| + |N| + 2 = (|P| + 3) + |N| + 2 = |P| + |N| + 5$ tetrahedra.

Let us describe a construction which only requires four more tetrahedra. Consider the 2-tetrahedron 3-vertex triangulation of S^3 in figure 23 below. Take the connected sum of P with S^3 by removing normal neighborhoods of the vertex v_1 in S^3, and v in P. This vertex v_1 as the nice property that its induced 3-valent graph G is 1-edge-connected. For future reference, we will say that v_1 is a **good vertex**. It is not hard to see that a vertex A is a good vertex if it is the end point of two edges forming a good cone.

FIGURE 22. A subdivision of a tetrahedron in P.

FIGURE 23. A 3-vertex 2-tetrahedron triangulation of S^3.

$P \# S^3$ is made of the tetrahedra from P, the tetrahedra from S^3, plus one extra tetra-hedron. In fact, when there is a good vertex in the triangulation of one of two 3-manifolds, there is only one red vertex that needs to be added to G_1 and G_2 to obtain an orientation reversing isomorphism between G_1' and G_2'. Hence, we obtain a triangulation of P with 3 more tetrahedra and 2 vertices. What we need to notice here is that one of the two vertices of the new triangulation of P is also a good vertex (vertex v_3 in figure 23). Hence, $|P_{new} \# N| = |P_{new}| + |N| + 1 = |P| + 3 + |N| + 1 = |P| + |N| + 4$.

FIGURE 24. Only one red vertex is needed to obtain G_1' and G_2'.

We summarize the above constructions. If P has two vertices, then we can take the connected sum of P and N by inserting 2 tetrahedra unless P has a good vertex in which case the insertion of only 1 tetrahedron is necessary. If both P and N have one vertex, then $P \# N$ requires inserting 4 tetrahedra (we will see later that if either P or N has a good vertex then this construction requires the insertion of 2 tetrahedra only).

4.3. Construction. Let M_1, ..., M_k be closed orientable 3-manifolds with at least 2 vertices in each of their triangulations. Then there exists a triangulation for $M_1 \# ... \# M_k$ with $\sum |M_i| + k + 2$ tetrahedra and at least 2 vertices.

4.4. Construction. Let M be a triangulated closed orientable 3-manifold with $|M| = n$ and with exactly one vertex. Then, for $k \geq 3$, there exists a k-vertex triangulation for M with $(n + k + 1)$ tetrahedra.

4.5. Construction. Let M be a triangulated closed orientable 3-manifold with $|M| = n$ and with exactly k vertices. Then, there exists a $(k + 1)$-vertex triangulation of M with $(n + 2)$ tetrahedra. If M has a cone, this construction can be done by adding a single tetrahedron, and hence, obtaining a $(k + 1)$-vertex triangulation of M with $(n + 1)$ tetrahedra.

4.6. Construction. Let M_1, ..., M_k be closed orientable 3-manifolds with 1-vertex triangulations, and $k \geq 3$. Then there exists a triangulation for $M_1 \# ... \# M_k$ with $\sum |M_i| + 2k$ tetrahedra and 1 vertex.

We show in [2] that construction 4.6 is the most efficient way of taking connected sums.

4.7. Construction. Let M be a triangulated closed orientable 3-manifold with $|M| = n$ and at least two vertices. Then there exists a triangulation for $M \# L(3, 1)$ with (n + 2) tetrahedra.

Let M' be the manifold obtained from M after removing the link of a vertex. Without loss of generality, we assume M' contains a truncated tetrahedra as in figure 14. Let T be the triangulation of $\partial M'$, and G its dual. Let e be the dual of the thickened edge from figure 14. We insert 2 prisms along the shaded face, and we insert tips of tetrahedra so that the new dual G' of $\partial M'$ correspond to figure 25 below.

FIGURE 25. The dual, G', of the new cell decomposition of $\partial M'$.

FIGURE 26. A cell decomposition of a punctured $L(3, 1)$ and the dual G_1 of its boundary.

Since G' is planar, it is the dual of a cell decomposition of a 2-sphere, and hence, we did not change the topology of M'. Consider the cell decomposition N' of the punctured $L(3, 1)$ in figure 26 with the graph G_1 corresponding to the dual of $\partial N'$. We cut along

the face labeled 4 in figure 25, and we insert truncated tips so that the resulting graph G'_1 of G_1 is colored homeomorphic to G'. It is now clear that the union of M' and N' gives a well-defined triangulation of $M \# L(3,1)$ with $(n+2)$ tetrahedra.

4.8. Construction. Let M be a triangulated closed orientable 3-manifold with $|M| = n$ and at least two vertices. Then there exists a triangulation for $M \# \mathbf{RP}^3$ with $(n + 2)$ tetrahedra.

FIGURE 27. The new triangulation G' of G.

FIGURE 28. A cell decomposition of a punctured $\mathbf{R}P^3$ and the dual G_1 of its boundary.

Let M' be the manifold obtained from M after removing the link of a vertex. Without loss of generality, we assume we can find a truncated tetrahedron as in figure 14. Let T be the triangulation of $\partial M'$, and G its dual. Let e be the thickened edge from Figure 14. We insert 4 prisms along the shaded face and we insert tips of tetrahedra so that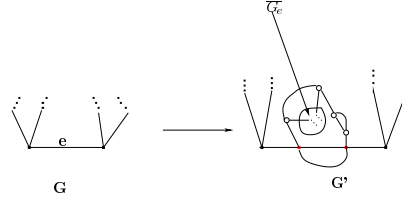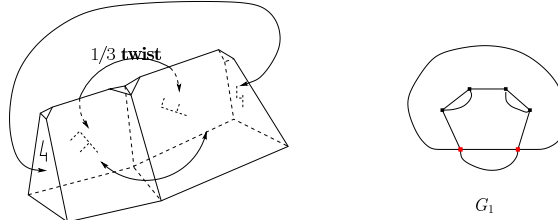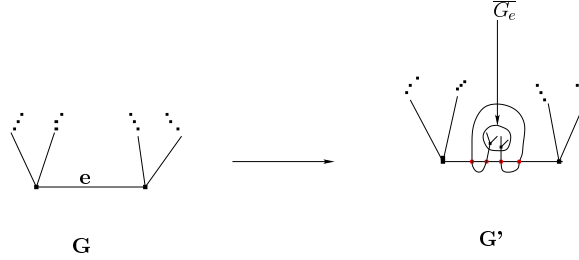 the resulting graph G' of G correspond to figure 27 above. Note, G' is planar and so we did not changed the topology of M'. Consider now the cell decomposition N' of the punctured $\mathbf{RP}^3$ in figure 28.

We cut along the face labeled 4 in figure 28, and we insert truncated tips so that the resulting graph G'_1 of G_1 is colored homeomorphic to G'. It is now clear that the union of M' and N' gives a well-defined triangulation of $M \# \mathbf{RP}^3$ with $(n+2)$ tetrahedra.

4.9. Construction. Let M be a triangulated closed orientable 3-manifold with $|M| = n$ and at least three vertices. Then there exists a triangulation for $M \# (S^1 \times S^2)$ with $(n + 2)$ tetrahedra. If M contains a good vertex, this construction can be done with one extra tetrahedron only.

4.10. Proposition. *There exists a 1-vertex triangulation of any closed orientable 3-manifold.*

Proof : Let M be a closed orientable reducible 3-manifold equipped with a triangulation. Suppose M has more than 1 vertex. Consider the 1-vertex 1-tetrahedron triangulation of S^3. Then $M\#S^3$ has $(t+3)$ tetrahedra and $(v-1)$ vertices by our Construction 4.1. We can repeat this construction $(v-2)$ times and obtain a manifold homeomorphic to M with $(t+3(v-1))$ tetrahedra and 1-vertex only. Q.E.D.

5. Small normal 2-Spheres in Minimal Triangulations

Using Theorem 2.1 and the constructions in the previous section, we show that minimal triangulations of reducible 3-manifolds contain *small* non-trivial normal 2-spheres. By ***small***, we mean normal 2-spheres which have quadrilateral types in not more than 2 tetrahedra. This will be essential in Andrew Casson's algorithm to check if a minimal triangulation is reducible or not.

5.1. Definition. A 3-manifold M is said to have a ***minimal triangulation*** τ if τ contains the smallest number of tetrahedra over all possible triangulations of M. By abuse of language, we say that M is minimal.

We say that a normal surface S has n ***quadrilateral types*** $(<S>=$ n) if there exist exactly n tetrahedra with the property that S intersects each of these tetrahedra in quadrilaterals. Note, if S and T are two compatible normal surfaces with $<S>=$ n and $<T>=$ m, then $(n+m) \geq <S+T> \geq$ max(n, m). Moreover, normalizing an embedded surface decreases its weight but may increase its number of quadrilateral types. Given a normal surface F with normal vector x_F, let $q_1, ..., q_{4t}$ be the entries in x_F corresponding to the quadrilaterals of F. We denote by $\#_{\mathbf{q}}(\mathbf{F})$, the sum of the q_i's.

To prove the existence of small normal 2-spheres in minimal triangulations, we first show the existence of a normal 2-sphere with the property that all the collapsing surfaces are inessential. If we find such a 2-sphere, then collapsing it will result in a decomposition of M in exactly two summands. Because M is minimal, we will use the constructions in the previous section to conclude that S cannot have more than 2 quadrilateral types. To show the existence of a non-trivial normal 2-sphere with no essential collapsing surfaces, we use a result due to W. Jaco and J. Tollefson [11].

Theorem 4.1 ([11]): A normal two-sphere F is a vertex surface if and only if F has the property that whenever there exists an annulus A which is an exchange surface for F then the two disjoint disks in F bounded by ∂A are normal isotopic.

Here, an ***exchange surface*** A for F is a surface with the following properties: 1) $fr(A) = A \cap F$, 2) A has an orientable regular neighborhood $N(A)$, and 3) for every tetrahedron Δ, each component of $\Delta \cap A$ is a 0-weight disk L spanning two distinct elementary disks E_1, E_2 of F such that $\partial L = L \cap (E_1 \cup E_2 \cup \partial\Delta)$ and $L \cap E_i$ is an arc joining the interiors of two distinct 2-faces of Δ.

We see from this definition that if A is a collapsing annulus or a collapsing Mobius band, then we can either push A off the 2-skeleton (if A is an annulus) or look at $\partial Nbhd(A)$ (if A is a Mobius band) to find an annulus which is an exchange surface. Hence, if F is a vertex 2-sphere, we now know that any collapsing annulus or Mobius band is inessential. On the other hand, Jaco and Tollefson's theorem does not say anything about collapsing disks. To go around this problem, we define a complexity Q. Let S be a normal surface. Then $\boldsymbol{Q(S)}$ represents a pair, ordered lexicographically, whose first and second entries are $<S>$ and $\#_q(S)$ respectively: Q(S) = $(<S>, \#_q(S))$.

We are now ready to prove the Lemma for the main theorem.

5.2. Lemma. *Let M be a triangulated closed orientable 3-manifold which contains a non-trivial normal 2-sphere. Then M contains a non-trivial normal 2-sphere whose collapsing surfaces are all inessential.*

Proof. Consider the set of non-trivial normal 2-spheres. This set is non-empty by assumption. In this set, choose the 2-sphere F which is minimal with respect to Q. We want to show that such a 2-sphere does not have any essential collapsing surfaces, but first, we want to show that it is a vertex surface.

Suppose F is not a vertex surface, i.e. suppose x_F is not a vertex of $P(M,\ T)$, i.e. suppose for all positive integer k, $k \cdot x_F$ is not a vertex solution. Let k be the smallest positive integer such that $k \cdot x_F = S$ is an integral solution (and, by assumption, not a vertex solution). Then $\chi(S)$ must divide $\chi(F)$, and so $\chi(S) = 1$ or 2.

Case 1: $\chi(S) = 2$. Since F is an integral multiple of S, $S = F$. Suppose there exists a positive integer n such that $nF = V_1 + ... + V_k + W_1 + ... + W_r + X$, where at least one of the summands is not an integral multiple of F. Without loss of generality, we can assume that the V_i's are 2-spheres, the W_i's are real projective planes, and X is a (possibly empty or disconnected) surface with non-positive Euler characteristic. Because the Euler characteristic is preserved under surface addition, we have $2k + r \geq 2n$.
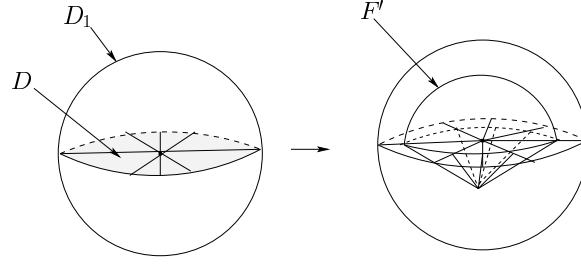
Subcase 1: Suppose that $r = 0$. Then $nF = V_1 + ... + V_k + X$ with $k \geq n$. Consider the 2-sphere, say V_1, which has the smallest $\#_q$ over all the V_i'. Since the surface addition preserves the quadrilateral types and the number of quadrilaterals for each quadrilateral type, we have $< V_1 > \leq < F >$. But by assumption, $< F > \leq < V_1 >$. Hence, $< F > = < V_1 >$. It is crucial to notice that, not only F and V_1 have the same number of quadrilateral types, but they must also have their quadrilaterals in the same tetrahedra. Moreover, $n \cdot (\#_q(F)) \geq k \cdot (\#_q(V_1)) + \#_q(X)$. Since $k \geq n$ and $\#_q(V_1) \geq \#_q(F)$, the only way for the inequality to be true is if it is an equality and $k = n$ and $\#_q(X) = 0$. We conclude that X was actually empty. Also, $\#_q(V_1) \leq \#_q(F)$. By assumption, $\#_q(F) \leq \#_q(V_1)$, so $\#_q(F) = \#_q(V_1)$. Therefore, V_1 is a parallel copy of F. We remove V_1, X, and a copy of F from the equation to obtain $(n-1)F = V_2 + ... + V_n$. We repeat the argument for the next 2-sphere, say V_2, which as the smallest $\#_q$. We conclude that V_i is a parallel copy of F for $1 \leq i \leq n$. This contradicts the fact that F was not an integral solution.

Subcase 2: Suppose $r \neq 0$. We look at the equation $2nF = 2V_1 + ... + 2V_k + 2W_1 + ... + 2W_r + 2X$, where $2V_i$ represents two copies of a 2-sphere and $2W_j$ represents a 2-sphere. We repeat the same argument as in Subcase 1 to conclude that each V_i is a parallel copy of F, that each $2W_i$ is a parallel copy of F, and that X is empty. Since, say $2W_1$ is parallel to F, we conclude that F is the double of the projective plane W_1. This contradicts the fact that k was the smallest positive integer such that S $(= F)$ is an integral solution.

Case 2: $\chi(S) = 1$. Since F is an integral multiple of S, $F = 2S$. Suppose there exists a positive integer such that $nS = V_1 + ... + V_k + W_1 + ... + W_r + X$, where the V_i's are 2-spheres, the W_j's are real projective planes, and $\chi(X) \leq 0$. We look at the new equation $nF = 2nS = 2V_1 + ... + 2V_k + 2W_1 + ... + 2W_r + 2X$. We run the same argument as in Subcase 1 of Case 1 to conclude that each V_i and each $2W_j$ is a parallel copy of F, and that X is empty. Hence, each V_i and $2W_j$ represents the double of S. Therefore, each summand of the equation is an integral multiple of S. We conclude that S is a vertex solution, or equivalently, F is a vertex surface. Contradiction.

Using the above theorem from Jaco and Tollefson, we know that if F is a non-trivial normal 2-sphere which minimizes Q, then F does not have any essential collapsing annuli or Mobius bands.

Let D be an essential collapsing disk. We construct another non-trivial normal 2-sphere with a smaller number of quadrilaterals: let D_1 and D_2 be the two disjoint disks on F such that $D_1 \cap D_2 = \partial D$. Consider the 2-sphere $D \cup D_1$. By assumption, this 2-sphere does not bound a ball whose interior is disjoint from F. We push D off the vertex and we obtain a new normal 2-sphere F'. See figure 29 below. Note, F' is represented by the same set of quadrilaterals as D_1. Because $\#_q(F)$ is minimal, we conclude that D_2 is composed of triangles only. Hence, $D \cup D_2$ bounds a ball. This is a contradiction and so D was inessential after all.

FIGURE 29. D is pushed off the 2-skeleton.

$\square$

Remark. Let A be a collapsing surface. We now know that the two disjoint disks D_1 and D_2 on F such that $\partial A = \partial D_1 \cup \partial D_2$ are normal isotopic. Moreover, Lemma 4.6 of [11] tells us that the ball with boundary $A \cup D_1 \cup D_2$ whose interior is disjoint from F does not contain any vertex from the triangulation. This is a crucial part of the proof of the following theorem.

5.3. Theorem. *Let M be a closed orientable 3-manifold equipped with a minimal triangulation. If M contains a non-trivial normal 2-sphere, then M contains such a 2-sphere with at most 2 quadrilateral types.*

By Lemma 5.2 we know there exists a non-trivial normal 2-sphere S with only inessential collapsing surfaces. If $< S > \leq 2$, then we are done. So we will assume that $< S > \geq 3$ and we will contradict the minimality of M. Consider the 2-sphere F constructed in Lemma 5.2. We collapse S. Since S has no essential collapsing surfaces, we end up with a decomposition of M into exactly 2 summands:

(1) $M \cong M_1 \# M_2$.
(2) $M \cong M_1 \# (S^1 \times S^2)$
(3) $M \cong M_1 \# L(3,1)$.
(4) $M \cong M_1 \# \mathbf{R}P^3$.
(5) $M \cong M_1 \# S^3$.
(6) $M \cong L(3,1) \# L(3,1)$.
(7) $M \cong L(3,1) \# \mathbf{R}P^3$.
(8) $M \cong L(3,1) \# S^3$.
(9) $M \cong \mathbf{R}P^3 \# \mathbf{R}P^3$.
(10) $M \cong \mathbf{R}P^3 \# S^3$.
(11) $M \cong S^3 \# S^3$.
(12) $M \cong L(3,1) \# (S^1 \times S^2)$.
(13) $M \cong \mathbf{R}P^3 \# (S^1 \times S^2)$.
(14) $M \cong S^3 \# (S^1 \times S^2)$.

By the above remark, the sum of the number of vertices in the resulting triangulated summands of M must be at least the number of vertices in the original triangulation. Hence, we can eliminate (6), (7), (9), (12), and (13).

Jaco and Rubinstein [9] showed that the minimal triangulations of S^3, $\mathbf{R}P^3$, $S^1 \times S^2$, and $L(3,1)$ have less than 3 tetrahedra. Hence, any non-trivial normal 2-sphere in these triangulations has less than 3 quadrilateral types. We can thus eliminate (8), (10), (11), (14).

(5) clearly contradicts the minimality of M.

In (3) and (4), M_1 must contain at least the same number of vertices as M plus one from the collapsing of S. Hence, M_1 must have at least 2 vertices, and constructions 4.7 and 4.8 can be used to obtain a new triangulation of M with less tetrahedra. Contradiction.

For similar reasons, constructions 4.1 and 4.9 can be used in (1) and (2) to construct triangulations of M with less tetrahedra. Contradiction. This proves the theorem.   □

6. Casson's Algorithm to Decompose a Closed Orientable 3-Manifold into Irreducible Pieces

In 1952, Edwin E. Moise [16] proved that any 3-manifold can be triangulated and that, given any two triangulations K_1 and K_2 of the same 3-manifold, K_1 and K_2 are equivalent, i.e. they have isomorphic subdivisions. As of today, it has actually been proven that we can obtain K_2 from K_1 through a series of Pachner moves [17], though no bounds on the number of moves has been found for arbitrary 3-manifolds [1].

Let M be a closed orientable 3-manifold triangulated with t tetrahedra. We mention that, even though the following lemmas were proved by the author, the actual results are due to Andrew Casson.

6.1. Definition. Let $\overline{N}(\mathrm{M})$ be the set of surfaces in M which intersect each tetrahedron in triangles or quadrilaterals. Precisely, $\overline{N}(\mathrm{M})$ is the set of surfaces which satisfy the normal surface equations but may not satisfy the quadrilateral property.

A type w is a function which assigns, to each tetrahedron, one of the 3 possible types of quadrilateral. Let N(M) be the set of normal surfaces and let $N_w(\mathrm{M})$ be the set of normal surfaces of type w. Note that $N(M) = \bigcup_w N_w(M)$.

The reason we look at $\bigcup_w N_w(M)$ instead of $N(M)$ is because the solution space of $N(M)$ does <u>not</u> form a cone in $\mathbf{R}^{7t}$ whereas each $N_w(M)$ does form a cone. Indeed, if S_1, $S_2 \in N(M)$ are not of the same type, their normal sum do not represent an embedded surface.

6.2. Claim. There exists a 2-sphere in $N(M)$ if and only if there exists a surface S in $N_w(M)$ for some w, with $\chi(S) > 0$.

<u>**Proof:**</u> $\Rightarrow$ Trivial since $N(M) = \bigcup_w N_w(M)$.
   $\Leftarrow$ If there is a surface S in $N_w(M)$ for some w, then this surface must be embedded. Since $\chi(S) > 0$, one of the connected components of S must be homeomorphic to either a $\mathbf{RP}^2$ or a 2-sphere. If it is a 2-sphere, then we are done. If it is a $\mathbf{RP}^2$, then a regular neighborhood of this projective plane is homeomorphic to a twisted I-bundle over it. This twisted I-bundle is homeomorphic to a punctured $\mathbf{RP}^3$. Hence, $\partial N(RP^2) \cong S^2$ which has the desired property.   □

Suppose M has a unique vertex v. Let $t_1, t_2, ..., t_{4t}$ be the triangle coordinates corresponding to v. Let $N_{w,i}(M) = \{S \in N_w(M) | t_i = 0\}$

6.3. Claim. There exists a non-trivial 2-sphere in $N_w(M)$ if and only if there exists a surface S in $N_{w,i}(M)$ for some w, i, and with $\chi(S) > 0$.

<u>**Proof:**</u> $\Rightarrow$ If S^2 is non-trivial, then it cannot be the boundary of a neighborhood of a vertex and hence, we must have $t_i = 0$ for some i.
   $\Leftarrow$ if $S \in N_{w,i}(M)$ then S cannot be the boundary of a vertex neighborhood and hence, cannot be trivial. As in claim 6.2, $\chi(S) > 0$ implies the existence of a non-trivial S^2.   □

We define $\mathbf{C_{w,i}(M)}$ to be the solution space to the surface equations with type w, $t_i = 0$ for a fixed i, and the rest of the variables being *real* non-negative.

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[1]Explicit bounds have been found for Seifert fibered spaces in [15]

Remark: let us define the Euler characteristic function on a rational-entry vector satisfying the surface equations and the quadrilateral property. Let $S = (a_1, a_2, ..., a_{7t})$ be an integer solution of the surface equations representing an embedded surface. Let a_1, ..., a_{4t} and a_{4t+1}, ..., a_{7t} denote the respective coefficients of the triangle and quadrilateral entries. We then define χ in the following way:

$$\chi(S) = [\sum_{i=1}^{4t} a_i \cdot (\frac{1}{d_{i_1}} + \frac{1}{d_{i_2}} + \frac{1}{d_{i_3}}) - \sum_{i=1}^{4t} a_i \cdot (\frac{3}{2}) + \sum_{i=1}^{4t} a_i] +$$
$$[\sum_{i=4t+1}^{7t} a_i \cdot (\frac{1}{e_{i_1}} + \frac{1}{e_{i_2}} + \frac{1}{e_{i_3}} + \frac{1}{e_{i_4}}) - \sum_{i=4t+1}^{7t} a_i \cdot (\frac{4}{2}) + \sum_{i=4t+1}^{7t} a_i]$$
$$\chi(S) = [\sum_{i=1}^{4t} a_i \cdot (\frac{1}{d_{i_1}} + \frac{1}{d_{i_2}} + \frac{1}{d_{i_3}} - \frac{1}{2})] + [\sum_{i=4t+1}^{7t} a_i \cdot (\frac{1}{e_{i_1}} + \frac{1}{e_{i_2}} + \frac{1}{e_{i_3}} + \frac{1}{e_{i_4}} - 1)], \text{ where}$$

the d_i's and the e_j's represent the respective degrees of the edges touching the triangle i and the quadrilateral j.

Now let V be a rational solution of the surface equations, satisfying also the quadrilateral property ($V = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, ..., \frac{a_{7t}}{b_{7t}})$). Let $N = \Pi_{i=1}^{7t} b_i$. We then define χ as follows: $\chi(V) = \frac{1}{N}\chi(N \cdot V)$, where $N \cdot V$ denotes the integer solution $(\frac{a_1 \cdot N}{b_1}, \frac{a_2 \cdot N}{b_2}, ..., \frac{a_{7t} \cdot N}{b_{7t}})$. From this definition, it follows that χ is a linear function. Indeed, let V_1 and V_2 be two rational solutions. Then, $\chi(V_1) + \chi(V_2) = \frac{1}{MN}\chi(\frac{1}{MN}(V_1 + V_2)) = \frac{1}{MN}\chi(MN \cdot V_1) + \frac{1}{MN}\chi(MN \cdot V_2) = \frac{1}{M}\chi(M \cdot V_1) + \frac{1}{N}\chi(N \cdot V_2) = \chi(V_1) + \chi(V_2)$.

6.4. Claim. There exists a surface S in $N_{w,i}(M)$ for some w, i with $\chi(S) > 0$ if and only if there exists a vector V in $C_{w,i}(M)$ for some w, i, such that $\chi(V) > 0$, V is on an external ray of the cone $C_{w,i}(M)$, and $V \in \mathbf{Z}^{7t}$.

Proof : $\Leftarrow$ Suppose $\exists V \in C_{w,i}(M)$, $\chi(V) > 0$, and $V \in \mathbf{Z}^{7t}$. Then , trivially, $V \in N_{w,i}(M)$.

$\Rightarrow$ If $S \in N_{w,i}(M)$, then $S \in C_{w,i}(M)$. Suppose now that S is not on some external ray, i.e. suppose that $S = V_1 + V_2 + ... + V_n$ where $V_j \in C_{w,j}(M)$, and the V_j's are connected and they lie on an external ray. Since S has integer entries, we can assume without loss of geneality that the V_j's have rational entries. Then $\chi(S) = \chi(V_1 + ... + V_n) = \chi(V_1) + ... + \chi(V_n) > 0$. This implies that $\chi(V_j) > 0$ for some j. We multiply each entry of V_j be the least common multiple of the denominator of the entries. Since $\chi(V_j) > 0$, we conclude that either V_j is homeomorphic to a 2-sphere or a projective plane. If it is a projective plane, then we double each entry in the vector representation of the surface to obtain a 2-sphere on the same external ray. Also, V_j cannot represent a trivial 2-sphere because $i = 0$ for some i. V_j or its double gives us the desired surface. $\square$

6.5. Definition. $S \subset M$ is called *almost* [2] *normal* (resp. *almost normal*) if it is normal and if there exists at most one tetrahedron τ in which S intersects τ in triangles and octagons (resp. one octagon) only.

Define $A_{w,i,l}(M)$ to be the set of surfaces of type w, with $t_i = 0$, and one type of octagon being allowed only in the lth tetrahedron.

6.6. Claim. $\exists$ an almost normal $S^2 \Leftrightarrow \exists S \in A_{w,i,l}(M)$ for some w, i, l with $\chi(S) - o(S) > 0$, where o(S) is the number of octagons in S in the lth tetrahedron.

Proof : $\Rightarrow$ If there is an almost normal S^2, then clearly $S \in A_{w,i,l}(M)$ for some w, i, l. Since S is almost normal, then $o(S) = 1$. This implies that $\chi(S) - o(S) = 1 > 0$.

$\Leftarrow$ If there is a S in $A_{w,i,l}(M)$ with $\chi(S) - o(S) > 0$, then, by definition, S is almost[2] normal. Since $\chi(S) - o(S) > 0$, and o(S)$\geq$ 1 (if o(S)=0 then S is normal non-trivial, but we already have cut along all such surfaces), then $\chi(S) > 1$. If $\chi(S) = 2$, then $o(S) = 1$ and we have an almost normal S^2. If $\chi(S) > 2$, then S must have an S^2

component with $\chi(S^2) - o(S^2) > 0$. This again implies that $o(S^2) = 1$ and we have an almost normal S^2. □

The algorithm :

Let M be a closed orientable 3-manifold equipped with a t-tetrahedra one-vertex triangulation. The set of surface equations is completely determined by the triangulation, and so are the cones $C_w(M)$. There are 3^t of them, one for each type w.

The first task of the procedure is to find a non-trivial normal 2-sphere. We use claim 6.4 to find one. Given a cone $C_{w,i}(M)$, we look at the set $A = C_{w,i}(M) \bigcap \{\sum_{i=1}^{7t} t_i = 1\}$. This set is a convex compact polyhedron with vertices having rational entries. On this polyhedron, we maximize the Euler characteristic function defined above. Linear programming theory tells us that this function attains its maximum at a vertex of this polyhedron and hence, on an external ray of the cone $C_{w,i}(M)$. There are several methods to find such a maximum. Indeed, Schrijver [19] (Theorem 15.3 page 198) first proved the existence of a method to find the maximum of a linear function on a convex compact polyhedron with a running time polynomial in the size of a matrix B. In our context, B describes the surface equations and the non-negativity of the entries t_i. The size of B (as defined in Schrijver page 29) happens to be a polynomial with respect to t. In fact, $size(B) \le 182t^2$. This upper bound comes from the fact that B is a $(7t) \times (6t + 7t)$ matrix with integer entries smaller than 2. See [12] for an explicit algorithm.

This vertex, on which χ is maximum, has rational entries and so we multiply it by the least common multiple of the denominator of the entries. This new vector S with integer entries can have Euler characteristic 0, 1, or 2:

$\chi(S) = 1$: S represents a projective plane. We look at the surface $2S$ which must represent a non-trivial 2-sphere since S is non-trivial.

$\chi(S) = 2$: S is a non-trivial 2-sphere.

If $\chi(S) < 0$, then by claim 6.3 and claim 6.4 there are no non-trivial S^2 and the procedure stops here.

$\chi(S) > 2$ would contradict the facts that S is on an external ray of $C_{w,i}(M)$.

Once the procedure finds a non-trivial 2-sphere (S or $2S$), it collapses it using theorem 2.1. After cutting and collapsing along all non-trivial S^2's, the procedure needs to perform one more task. It needs to check if some of the resulting summands are homeomorphic to S^3.

The second task of the procedure is to find an almost normal 2-sphere. To do that, we use the Thompson-Rubinstein theorem which states that one of the resulting pieces is homeomorphic to S^3 if and only if there exists an almost normal 2-sphere in it. To find such a 2-sphere, we maximize the linear function $\chi(\bullet) - o(\bullet)$ over the set $A_{w,i,l}(M) \bigcap \{\sum_{i=1}^{7t} t_i = 1\}$. This gives us a solution on a rational vertex W, which in turn gives us an integer vector V. If we obtain $\chi(V) - o(V) \le 0$, then by claim 6.6 there are no almost normal 2-spheres and M_i is not homeomorphic to S^3 and the procedure stops. If $\chi(V) - o(V) = 1$, then M_i is homeomorphic to S^3. Note, $\chi(V) - o(V) > 1$ would imply that $\chi(V) > 2$ which would mean that V is disconnected which is impossible since V is a vertex.

Before calculating the complexity of the algorithm, we need to show that the procedure terminates after finitely many steps.

Fact 1: The procedure terminates after cutting along and collapsing finitely many non-trivial 2-spheres. This follows from Theorem 2.1 since cutting along a non-trivial S^2 strictly reduces the original number of tetrahedra in M.

Fact 2: If there are no non-trivial S^2, the procedure terminates. Indeed, claim 6.4 tell us that there are no such 2-sphere if and only if $\chi(S) \le 0$.

Fact 3: The procedure cannot be looking indefinitely for an almost normal S^2 if there are none. Indeed, if $\chi(V) - o(V) \le 0$, then by claim 6.6 there are no almost normal 2-spheres.

We summarize the steps needed in the algorithm to decompose a closed orientable 3-manifold into irreducible pieces. Let M be given by a t-tetrahedra and v-vertex triangulation.

Step 1: Construct a (possibly disconnected) normal surface S obtained by normalizing the boundary of a regular neighborhood of a maximal tree of the 1-skeleton. We collapse S using Theorem 2.1. This may result in a decomposition of M: $M \cong M_1 \# M_2 ... \# M_k$ $\# r_1(S^1 \times S^2) \# r_2 \mathbf{RP}^3 \# r_3 L(3,1)$. For each summand not homeomorphic to $\mathbf{RP}^3$, $S^1 \times S^2$, S^3, or $L(3,1)$, and having more than one vertex in its triangulation repeat step 1. We end up with a decomposition of M where each summand is either $\mathbf{RP}^3$, $S^1 \times S^2$, $L(3,1)$, or it has a 1-vertex triangulation. Let us call M one of the 1-vertex summands. We will repeat the procedure for each of the other 1-vertex summands.

Step 2 : Go through each cone $C_{w,i}(M)$ to find a non-trivial 2-sphere. If no such sphere is found, go to step 3. If one is found, then collapse it using Theorem 2.1. We obtain a further decomposition of M. For each new summand having more than 1-vertex in its triangulation, repeat step 1. For each new summand having a 1-vertex triangulation, repeat step 2.

Step 3: Go through each cone $A_{w,i,l}(M)$ to find an almost[2] normal 2-sphere. If none are found, go to step 4. If one is found, then we know the summand is homeomorphic to S^3.

Step 4: We obtain a decomposition of M where each summand is not homeomorphic to S^3 and is either irreducible with a 1-vertex triangulation, or is homeomorphic to $\mathbf{RP}^3$, $S^1 \times S^2$, or $L(3,1)$.

We now describe the complexity of the algorithm. Note, our goal here is not to give explicit bounds for the running time of this algorithm. Rather, we only make a distinction between polynomial and exponential running time.

Complexity of the algorithm:

1. How long does it take for the procedure to change an arbitrary triangulation of M into a one-vertex one?
2. How long does it take for the procedure to find a non-trivial normal S^2?
3. How long does it take for the procedure to collapse a non-trivial normal S^2?
4. How many non-trivial normal 2-spheres can there be?
5. How long does it take for the procedure to look for an almost normal S^2?

1. Suppose M has more than one vertex. We take the boundary of a regular neighborhood of an edge. The weight of that surface is bounded by twice the number of edges. Hence, normalizing this surface takes time linear in the number of tetrahedra. We obtain a union of normal 2-spheres. We collapse each of them. See (3.) for the running of the collapsings.

2. We have to look through each cone $C_{w,i}(M)$. There are at most $4t \cdot 3^t$ of them. In each cone, we maximize χ to find a 2-sphere. This can be done in polynomial time $O(t^n)$, where n is independent of t. Hence the running time of this step is $O(t^{n+1}) \cdot 3^t$.

3. This part of the procedure refers to Theorem 2.1. First, to count the number of $\mathbf{R}P^3$ summands from step 2, we calculate $H_2(M; \mathbf{Z}) using cell decomposition$. This can be done in polyniomal time in t by transforming a $(2t) \times t$ matrix in its row reduced echelon form (see [19]). It then suffices to compare the number of missing $\mathbf{Z}_2$ factors in the second homology of the resulting summands of M with the number of missing $\mathbf{Z}_2$ factors in the second homology of M. Next, we count the number of collapsing surfaces. Note, this number depends on the weight of S but we are only concerned with the surfaces which do not consist entirely of common sides of adjacent I-bundles or adjacent tips. There cannot be more than 4t such surfaces. Finally, it was shown in step 5 that the number of $S^1 \times S^2$ summands is $n + 2 - k$, where n is the number of collapsing surfaces which do not

       consist entirely of common sides of adjacent I-bundles or adjacent tips and k is
the number of resulting summands. Hence, it takes polynomial time to collapse
a normal 2-sphere and obtain the triangulations of the resulting pieces as in the
conclusion of Theorem 2.1.

4. Cutting along a non-trivial normal 2-sphere strictly reduces the number of tetra-
hedra, so there could not be more than t of them. In fact, it is shown in [3] that
there cannot be more than $\lfloor t \rfloor / 2$ of them.

5. We look through each cone$A_{w,i,l}(M)$. There are at most $3^t \cdot 4t \cdot t$ of them. In each
cone, we maximize $(\chi - o)$. As in 2., such a maximum is found in polynomial time
$O(t^m)$, where m is independent of t. The running time for this step is $O(t^{m+2}) \cdot 3^t$.

6.7. Theorem. *Let M be a closed orientable 3-manifold equipped with a minimal trian-
gulation with t tetrahedra. Then there is an algorithm to check if M is reducible or not,
and this algorithm runs in polynomial time with respect to t.*

    Proof : What makes the above algorithm run in exponential time is the number of
cones $C_{w,i}(M)$ (there are about 3^t of them) and $A_{w,i,l}$ (there are about 3^t of them). When
M is minimal though, not only we do not need to look through all the cones $C_{w,i}(M)$,
but we don't even need to run the Thompson-Rubinstein algorithm.

    If there are no non-trivial embedded normal 2-spheres, then a famous result of Kneser [13]
tells us that M does not contain any embedded essential 2-sphere. Hence, M is irreducible.
Suppose M does contain a non-trivial normal 2-sphere S. Let S be a normal 2-sphere
constructed in lemma 5.2. When we collapse it we end up with at most 2 summands for
M. Since M is minimal, none of the summands can be homeomorphic to a 3-sphere. This
tells us that S is essential. Hence, M is reducible. What is important to notice here is that
$< S >\leq 2$ by Theorem 5.3. Therefore, if M is minimal and reducible, then it contains an
essential normal 2-sphere with 1 or 2 quadrilateral types.

    Consider the space of normal surfaces represented by a family of type w, where w
assigns the value 0 for the quadrilateral types in every tetrahedra except one. We noted
earlier that surface addition may increase the number of quadrilateral types. Indeed, if
F_1 has exactly one quadrilateral type in the jth tetrahedra Δ_j, and F_2 has exactly one
quadrilateral type in the ith tetrahedra, then $F_1 + F_2$ has a quadrilateral type in Δ_j and
Δ_i. Hence this space does not represent a cone. Consider now the space of normal surfaces
represented by a family, $w_{(i)}$, of type w, where w assigns the value 0 for the quadrilateral
types in every tetrahedra except in the ith one. It is now easy to see that this space is
a cone in $\mathbf{R}^{7t}$. Every non-trivial normal surface in this cone has one quadrilateral type,
and conversely, every non-trivial normal surface with one quadrilaterl type belong to a
cone $C_{w_{(j)},i}(M)$ for some fixed i and j. For each tetrahedron, there are three different
quadrilateral types and so there are three $w_{(j)}$'s. For each type $w_{(j)}$, there are $4t$ choices
for the triangle entries t_i. Hence, there are $3 \cdot t \cdot (4t) = 12t^2$ different cones.

    Similarly, we define the space of normal surfaces having two quadrilateral types in ex-
actly two distinct tetrahedra. If two tetrahedra Δ_j and Δ_k are fixed, this space represents
a cone $C_{w_{(j,k)},i}(M)$. In fact, the union of these cones, over all possible pairs of tetrahedra,
represents the space of normal surfaces with one or two quadrilateral types. For each pair
of tetrahedra, $\{\Delta_j, \Delta_k\}$, there are $3 \cdot 3$ possible types $w_{(j,k)}$. There are $\binom{t}{2}$ possible pairs of
distinct tetrahedra. Hence, there are $3 \cdot 3 \cdot \binom{t}{2} \cdot (4t) = 36t\binom{t}{2}$ different cones $C_{w_{(j,k)},i}(M)$.

    We now describe the algorithm to check if a minimal closed orientable 3-manifold M
is reducible or not. Fix a cone $C_{w_{(j,k)},i}(M)$ and look at the convex polyhedron $A =
C_{w_{(j,k)},i}(M) \bigcap \{\sum_{i=1}^{7t} t_i = 1\}$. We maximize χ on A to obtain a vertex solution S. If
$\chi(S) > 0$, then there exists a 2-sphere in $C_{w_{(j,k)},i}(M)$ and the procedure stops here: M
is reducible. If $\chi(S) \leq 0$, there are no 2-spheres in the cone $C_{w_{(j,k)},i}(M)$. Repeat this
step with a new cone $C_{w_{(r,s)},i}(M)$. If no 2-spheres have been found in any of the cones

$C_{w_{(j,k)},i}(M)$, then M is irreducible and the procedure stops here. Note, we do not need to look through the cones $C_{w_{(j)},i}(M)$ since they lie in some cone $C_{w_{(j,k)},i}(M)$.

As we have seen in Casson's algorithm, it takes polynomial time to look for a 2-sphere in each convex polyhedra A. Since there are only $36t\binom{t}{2}$ cones of the form $C_{w_{j,k},i}(M)$, it will take polynomial time to check if M is reducible or not.

There is, though, a problem in decomposing a minimal 3-manifold in irreducible pieces in polynomial time. Indeed, after collapsing a non-trivial normal 2-sphere, we end up with 2 summands which unfortunately may not be minimal. See [2] for a solution to this problem.

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